where P_{sk} are real constants and $\varphi_s(x_s)$ are continuous and satisfy the inequalities.

$$\varphi_s(x_s)\operatorname{sign} x_s > 0 \quad \text{for } x_s \neq 0 \qquad (s = 1, 2, \ldots, n) \tag{31}$$

Let the coefficients of this system and the elements of some basis $\{\alpha_{s\theta}\}$ be related by expressions (8). We then draw the following conclusions on the basis of Theorem 4 and Corollary 2 of Theorem 7:

1) The system under consideration is absolutely stable if and only if all the roots of secular equation (13) have negative real parts.

2) Let fulfillment of the above assumptions concerning the right sides of system (30) imply that $\det \| p_{sk} \| \neq 0$. The zero solution of this system is then unstable for any chosen functions $\varphi_s(x_s)$ satisfying inequalities (31) if and only if: (a) there exists at least one root of Eq. (13) with a positive real part, or (b) there exist roots of this equation with real parts equal to zero such that the number of groups of solutions corresponding to these roots is smaller than their multiplicity.

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ON THE STABILITY OF TRIANGULAR LIBRATION POINTS IN THE ELLIPTIC RESTRICTED THREE-BODY PROBLEM

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The results of a study of the stability of the equilibrium position of a nonautonomous Hamiltonian system with two degrees of freedom are presented. The parametric resonance domain for the libration points is determined to within the first power of the eccentricity. Formulas for computing the characteristic exponents are derived. The resonance values of μ and e for which the libration points can be unstable inside the stability domains are determined.

1. Let us consider three material points which attract each other according to Newton's law. Let the points S and J of masses m_1 and m_2 move relative to their common center of mass O along Keplerian ellipses of eccentricity e. The third body moves in the plane of the bodies S and J without affecting their motion.

We know [1] that the differential equations of motion of the three-body problem have a particular solution which corresponds to the libration points: the three bodies form an equilateral triangle which rotates about the common center of mass of the bodies.

In the case of the circular problem ($e = 0$) fulfillment of the inequality

$$27\ \mu\,(1-\mu) < 1$$

$$\mu = m_2 \mid (m_1 + m_2) \qquad (0 < \mu \leqslant {}^1/_2)$$

ensures that the libration points are stable in the first approximation [1].

In [2] we showed that the triangular libration points are, in fact, stable for all values of μ in the stability domain in the first approximation except for the two values

$$\mu = \frac{15 - \sqrt{213}}{30} = 0.0135160\ldots, \qquad \mu = \frac{45 - \sqrt{1833}}{90} = 0.0242938\ldots$$

for which they are unstable (*).

The elliptic problem was investigated in [5–7]. In [5, 6] an asymptotic method is used to analyze stability in the first approximation for small values of the eccentricity. In [7] numerical calculations in the plane $\mu\ e$ are used to obtain the domains of stability in the first approximation for an arbitrary eccentricity $(0 \leqslant e < 1)$.

In the present paper we determine the parametric resonance domain to within the first power of the eccentricity, derive expressions for the characteristic exponents in terms of the coefficients of the characteristic equation, and specify the values of μ and e for which the libration points can be unstable in the domains of stability in the first approximation in the case of the nonlinear problem.

2. Let the origin $q_i = p_i = 0$ be the equilibrium position of the system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2) \tag{2.1}$$

Here H is a Hamiltonian of period 2π in the independent variable t; this Hamiltonian is analytic in the neighborhood of the point $q_i = p_i = 0$.

Let the linearized system be stable and let all of its multipliers be distinct. We can then assume that the Hamiltonian has been reduced (e.g. see [8]) to the form

$$H = {}^1/_2\lambda_1\,(q_1^2 + p_1^2) + {}^1/_2\lambda_2\,(q_2^2 + p_2^2) + H_3 + H_4 + \ldots \tag{2.2}$$

Here $\pm\, i\lambda_1$, $\pm\, i\lambda_2$ are the characteristic exponents of the linearized system; H_m is a homogeneous function of degree m in q_i, p_i and of the period 2π in t.

Further, if the condition

$$k_1\lambda_1 + k_2\lambda_2 \not\equiv 0 \qquad (\mathrm{mod}\ 1) \tag{2.3}$$

is fulfilled for the integers k_1 and k_2 satisfying the equation $|k_1| + |k_2| = 3$ or $|k_1| + |k_2| = 4$, then there exists [9] an analytic canonical transform 2π-periodic in t which reduces the Hamiltonian to the form

*) The treatment of the problem in paper [2] entails certain results of [3]. The proof of instability in the latter paper contains certain inaccuracies which were subsequently rectified in [4].

$$H = \lambda_1 r_1 + \lambda_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + O((r_1 + r_2)^{5/2}) \quad (2.4)$$
$$(2r_i = q_i^2 + p_i^2)$$

The coefficients c_{ij} in (2.4) do not depend on t.

If the quadratic form
$$c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2$$
is of fixed sign in the domain $r_1 \geqslant 0$, $r_2 \geqslant 0$, then the equilibrium position is formally stable [8, 10, 11]. Formal stability implies that Liapunov instability is not manifested provided functions H_m up to an arbitrarily large m are retained in expansion (2.2); moreover, if there are trajectories emerging from the origin, then motion along these trajectories is very slow.

Stability in the case where condition (2.3) is not fulfilled for nonnegative k_1 and k_2 is investigated in [4].

If condition (2.3) is violated for at least one pair of nonnegative integers k_1 and k_2 whose sum is equal to three, then the Hamiltonian for suitably chosen variables q_i, p_i becomes
$$H = a_{k_1, k_2} r_1^{1/2 k_1} r_2^{1/2 k_2} \sin(k_1 \varphi_1 + k_2 \varphi_2) + O((r_1 + r_2)^2) \quad (2.5)$$
where
$$q_i = \sqrt{2r_i} \sin \varphi_i, \qquad p_i = \sqrt{2r_i} \cos \varphi_i \qquad (i = 1, 2)$$

The equilibrium position is unstable for $a_{k_1, k_2} \neq 0$.

If condition (2.3) is not fulfilled for a pair of nonnegative integers k_1 and k_2 whose sum is equal to three, the Hamiltonian can be transformed into
$$H = c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + b_{k_1, k_2} r_1^{1/2 k_1} r_2^{1/2 k_2} \sin(k_1 \varphi_1 + k_2 \varphi_2) + H'(r_i, \varphi_i, t)$$
$$(H' = O((r_1 + r_2)^{5/2})) \quad (2.6)$$

Fulfillment of the inequality
$$| b_{k_1, k_2} | k_1^{1/2 k_1} k_2^{1/2 k_2} > | c_{20} k_1^2 + c_{11} k_1 k_2 + c_{02} k_2^2 | \quad (2.7)$$
means that the equilibrium position is unstable (*); it is formally stable if the function $H - H'$ is of fixed sign in the neighborhood of the equilibrium position.

3. Let us investigate the motion of the body P with the aid of Nechvile coordinates using the true anomaly ν as the independent variable. The origin coincides with the center of mass O; the Ox-axis is directed towards the body J. We choose our unit of length in such a way that the distance between the bodies S and J is equal to unity. The differential equations of motion of the body P are then of the form [1]
$$\frac{d^2 x}{d\nu^2} - 2\frac{dy}{d\nu} = \frac{1}{1 + e\cos\nu} \frac{\partial \Omega}{\partial x}, \quad \frac{d^2 y}{d\nu^2} + 2\frac{dx}{d\nu} = \frac{1}{1 + e\cos\nu} \frac{\partial \Omega}{\partial y} \quad (3.1)$$
where
$$\Omega = \frac{x^2 + y^2}{2} + W, \qquad W = \frac{1 - \mu}{SP} + \frac{\mu}{JP}$$
$$SP^2 = (x + \mu)^2 + y^2, \qquad JP^2 = (x + \mu - 1)^2 + y^2$$

It is easy to verify the fact that system (3.1) corresponds to motion with the Hamiltonian
$$H = \frac{1}{2}(p_x^2 + p_y^2) + p_x y - p_y x + \frac{e\cos\nu}{2(1 + e\cos\nu)}(x^2 + y^2) - \frac{1}{1 + e\cos\nu} W \quad (3.2)$$

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*) We assume that the quantity $k_i^{1/2 k_i}$ in inequality (2.7) is equal to unity for $k_i = 0$.

Here p_x, p_y are the generalized momenta corresponding to the coordinates x and y.

The solution corresponding to the triangular libration point for the system with Hamiltonian (3.2) is the equilibrium position

$$x_0 = {}^1/_2(1-2\mu), \quad y_0 = {}^1/_2\sqrt{3}, \quad p_{x_0} = -{}^1/_2\sqrt{3}, \quad p_{y_0} = {}^1/_2(1-2\mu) \qquad (3.3)$$

Let us substitute variables as follows:

$$x = x_0 + q_1, \quad y = y_0 + q_2, \quad p_x = p_{x_0} + p_1, \quad p_y = p_{y_0} + p_2$$

Solution (3.3) then corresponds to the equilibrium position $q_1 = q_2 = p_1 = p_2 = 0$. Expanding the Hamiltonian in powers of q_i, p_i and discarding the terms independent of q_i and p_i, we obtain

$$H = H_2 + H_3 + H_4 + \ldots \qquad (3.4)$$

where

$$H_2 = \frac{1}{2}(p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 + \frac{1}{8(1+e\cos\nu)} \times$$

$$\times [q_1^2 - 5q_2^2 - 6\sqrt{3}(1-2\mu)q_1q_2] + \frac{e\cos\nu}{2(1+e\cos\nu)}(q_1^2 + q_2^2) \qquad (3.5)$$

$$H_3 = \frac{1}{16(1+e\cos\nu)}[-7(1-2\mu)q_1^3 + 3\sqrt{3}q_1^2q_2 + 33(1-2\mu)q_1q_2^2 + 3\sqrt{3}q_2^3]$$

$$H_4 = \frac{1}{128(1+e\cos\nu)}[37q_1^4 + 100\sqrt{3}(1-2\mu)q_1^3q_2 - 246q_1^2q_2^2 -$$

$$- 180\sqrt{3}(1-2\mu)q_1q_2^3 - 3q_2^4]$$

4. The elliptic problem entails the possibility of parametric resonance. The boundaries of the instability domain for small eccentricities can be found by asymptotic methods. According to [8], parametric resonance occurs in the neighborhood of those values of μ for which the quantities λ_1 and λ_2 for $e = 0$ satisfy the relations

$$\lambda_1 = {}^1/_2 N, \qquad \lambda_2 = {}^1/_2 N, \; \lambda_1 + \lambda_2 = N \quad (4.1)$$

where N is an integer. From [2] we infer that $\lambda_1 = \omega_1$, $\lambda_2 = -\omega_2$ for $e = 0$. Here ω_1, ω_2 are the real positive roots of the equation

$$\omega^4 - \omega^2 + 27/4\mu(1-\mu) = 0 \qquad (4.2)$$

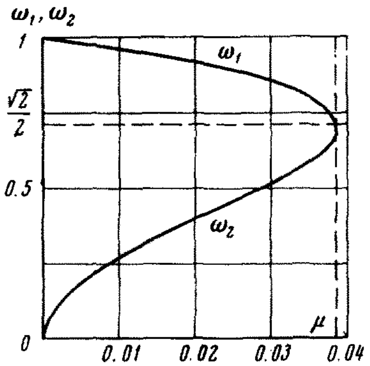

Fig. 1

Figure 1 shows ω_1 and ω_2 as functions of μ. Simple analysis shows that the only relation of (4.1) which is fulfilled in the stability domain of the circular problem is $\omega_2 = {}^1/_2$. Here

$$\mu = {}^1/_6(3 - 2\sqrt{2}) = 0.0285954\ldots$$

Computations by the averaging method show that the boundaries of the instability domain in the neighborhood of this value of μ to within the first power of the eccentricity are of the form

$$\mu = 0.0285954\ldots \pm e\, 0.0594720\ldots \qquad (4.3)$$

The stability and instability domains of the linearized problem for an arbitrary eccentricity e can be obtained by numerical calculation [7]. The stability domains of the linearized problem in the plane μe appear as the area indicated by grid lines in Fig. 2. The libration points lying outside these domains are unstable; the necessary conditions of stability are fulfilled for values of μe lying inside them.

5. It is possible for the motion to be unstable in the domains of the necessary stability conditions for values of the parameters μ, e which satisfy resonance relations (2.3) for nonnegative integers k_1, k_2. To find the

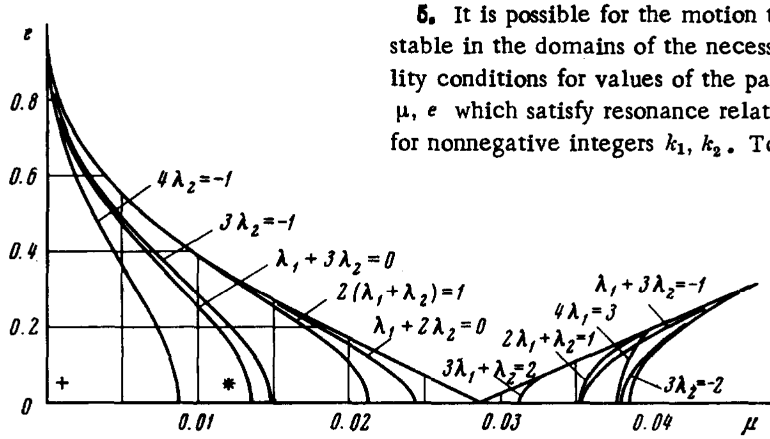

Fig. 2

resonance values of the parameters we first obtain expressions for the quantities λ_1, λ_2 in terms of the coefficients of the characteristic equation

$$\rho^4 - a_1\rho^3 + a_2\rho^2 - a_1\rho + 1 = 0$$

The coefficient a_1 is equal to the trace of the fundamental matrix of the linearized system computed for $\nu = 2\pi$; a_2 is equal to the sum of all its principal second-order minors. In the plane of coefficients a_1, a_2 the domain of stability of the linearized system is defined by the system of inequalities [12]

$$-2 < a_2 < 6, \qquad 4(a_2 - 2) < a_1^2 < {}^1/_4\,(a_2 + 2)^2$$

The roots of the characteristic equation in this domain can be written as

$$\rho_1 = e^{i2\pi\lambda_1}, \quad \rho_2 = e^{i2\pi\lambda_2}, \quad \rho_3 = e^{-i2\pi\lambda_1}, \quad \rho_4 = e^{-i2\pi\lambda_2}$$

It is easy to verify that

$$a_1 = 2(\cos 2\pi\lambda_1 + \cos 2\pi\lambda_2), \qquad a_2 = 2 + 4\cos 2\pi\lambda_1 \cos 2\pi\lambda_2$$

Hence, $\cos 2\pi\lambda_1$ and $\cos 2\pi\lambda_2$ satisfy the equation

$$z^2 - {}^1/_2\,a_1 z + {}^1/_4\,(a_2 - 2) = 0 \tag{5.1}$$

We cannot determine λ_1 and λ_2 unambiguously from this equation. Let us consider the limiting case of the circular problem in order to find the single-valued expressions for λ_1 and λ_2. For $e = 0$ the roots of Eq. (5.1) are

$$z_{1,2} = {}^1/_2\,(\cos 2\pi\omega_1 + \cos 2\pi\omega_2 \pm |\cos 2\pi\omega_1 - \cos 2\pi\omega_2|)$$

Making use of (4.2), we can readily verify the fact that $\cos 2\pi\omega_1 \geqslant \cos 2\pi\omega_2$. Hence,

$$\lambda_1 = (2\pi)^{-1}\,\mathrm{Arc}\cos z_1, \qquad \lambda_2 = (2\pi)^{-1}\,\mathrm{Arc}\cos z_2$$

Further, recalling that $1 \geqslant \omega_1 \geqslant {}^1/_2\sqrt{2} \geqslant \omega_2 \geqslant 0$, we obtain

$$\lambda_1 = 1 - (2\pi)^{-1}\arccos z_1 \qquad \text{for all} \quad \omega_1 \text{ and } \omega_2$$

$$\lambda_2 = \begin{cases} -(2\pi)^{-1}\arccos z_2 & \text{for } 0 \leqslant \omega_2 \leqslant {}^1/_2 \\ -1 + (2\pi)^{-1}\arccos z_2 & \text{for } {}^1/_2 \leqslant \omega_2 \leqslant {}^1/_2\sqrt{2} \end{cases}$$

We have thus eliminated the ambiguity in the determination of the quantities λ_1 and λ_2 and can compute them from the formulas

$$\lambda_1 = 1 - (2\pi)^{-1} \arc\cos {}^1/_4 (a_1 + \Delta) \quad \text{for } \mu \geqslant 0$$

$$\lambda_2 = \begin{cases} -(2\pi)^{-1} \arc\cos {}^1/_4 (a_1 - \Delta) & \text{for } 0 \leqslant \mu \leqslant {}^1/_6 (3 - 2\sqrt{2}) \\ -1 + (2\pi)^{-1} \arc\cos {}^1/_4 (a_1 - \Delta) & \text{for } \mu \geqslant {}^1/_6 (3 - 2\sqrt{2}) \end{cases} \tag{5.2}$$

$$\Delta = (a_1^2 - 4a_2 + 8)^{1/2}$$

To find the resonance values of the parameters we must compute the fundamental matrix for $\nu = 2\pi$ for fixed μ and e from the domain of the necessary conditions of stability, find the coefficients a_1, a_2 of the characteristic equation, and then compute λ_1 and λ_2 from formulas (5.2). We carried out these calculations by computer. The curves (see Fig. 1) on which the resonance relations are fulfilled are shown inside the stability domains in the plane μe. For $e = 0$ these curves are perpendicular to the $O\mu$-axis and emerge from points defined by the following data:

μ	0.0087...	0.0135...	0.0148...	0.0212...	0.0242...
	$4\lambda_2 = -1$	$\lambda_1 + 3\lambda_2 = 0$	$3\lambda_2 = -1$	$\lambda_1 + \lambda_2 = {}^1/_2$	$\lambda_1 + 2\lambda_2 = 0$
μ	0.0312...	0.0353...	0.0353...	0.0378...	0.0380...
	$3\lambda_1 + \lambda_2 = 2$	$2\lambda_1 + \lambda_2 = 1$	$\lambda_1 + 3\lambda_2 = -1$	$4\lambda_1 = 3$	$3\lambda_2 = -2$

For values of μe belonging to the resonance curves we can have either instability or formal stability. Formal stability is the only possibility outside the resonance curves. Resolution of the latter questions requires calculations based on the results of [4, 10, 11].

For the Sun-Jupiter and Earth-Moon systems we have $\mu = 0.000953...$, $e = 0.048253...$ and $\mu = 0.012116...$, $e = 0.054900...$, respectively. The points with these parameter values are marked + and • in Fig. 2. They do not fall on the resonance curves.

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AN OPTIMAL PROBLEM OF SATELLITE GUIDANCE BY MEANS OF A GYROSCOPE

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The use of gyroscopes for attitude cantrol and stabilization of space vehicles in the case of large angles is considered.

The simplest formulation of this nonlinear problem is investigated. An artificial earth satellite is equipped with a balanced two-axis gyro in a gimbal mount which acts as its final control element. The center of inertia of the gyro coincides with the center of inertia of the satellite body, and the axis of the outer gimbal (output axis) is parallel to one of the principal axes of inertia of the vehicle. It is assumed that the system is not acted on by external moments, so that its moment of momentum vector remains constant.

After stabilization of the angular position of the satellite on its orbit, i.e. after elimination of the initial angular velocities of the system, the entire moment of momentum is borne by the gyro wheel. The system can be rotated by altering the position of the gyro wheel axis (spin axis); the controls are the moments M_α and M_β acting on the gimbal axes. The angles of rotation α and β of the gimbals are called the "control angles".

Although the results obtained are largely qualitative in character, they can be used in conjunction with the iteration method to construct a more exact solution.

One of the two controls in the control mode just described, namely β, is varied relay fashion. The angle α, i.e. the rotation of the outer gimbal between the initial and final rapid rotations, is varied periodically and depends on the angle of nutation ϑ and on the inertial characteristic of the system.

During guidance the z-axis describes looped ($n < 0$) or wavy ($n > 0$) curves on a fixed unit sphere; these curves are bounded by two parallels for which $\sin\vartheta = \pm n$. The self-intersection points of the loops or the inflection points of the wavy curves correspond to $\vartheta = \vartheta_0$.

Let the initial position of the satellite body be known and let the purpose of control be to achieve a certain attitude change, i.e. let the final position of the vehicle in space be specified. As the spin axis rotates in the satellite body and in inertial space, the satellite body acquires an angular velocity in accordance with the law of conserva-